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Non-holonomic and semi-holonomic frames in terms of Stiefel and Grassmann tangent bundles

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Abstract

We prove that the bundles of non-holonomic and semi-holonomic second-order frames of a real or complex manifold M can be obtained as extensions of the bundle $F^2(M)$ of second-order jets of (holomorphic) diffeomorphisms of $(\mathbb{K}^n, 0)$ into M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $\dim_{\mathbb{K}}(M) = n$ and $\mathcal{F}(M)$ is the bundle of $\mathbb{K}$-linear frames of M we will associate to the tangent bundle $E = T(\mathcal{F}M)$ two new bundles $\mathcal{S}t_n(E)$ and $\mathcal{G}_n(E)$ with fibers of type the Stiefel manifold $\mathcal{S}t_n(V)$ and the Grassmann manifold $\mathcal{G}_n(V)$, respectively, where $V = \mathbb{K}^n \oplus \mathfrak{gl}(n, \mathbb{K})$. The natural projection of $\mathcal{S}t_n(E)$ onto $\mathcal{G}_n(E)$ defines a $\mathrm{GL}(n, \mathbb{K})$-principal bundle. We have found that the subset of $\mathcal{G}_n(E)$ given by the horizontal n-planes is an open sub-bundle isomorphic to the bundle $\hat{\mathcal{F}}^2(M)$ of semi-holonomic frames of second-order of M. Analogously, the subset of $\mathcal{S}t_n(E)$ given by the horizontal n-bases is an open sub-bundle which is isomorphic to the bundle $\tilde{\mathcal{F}}^2(M)$ of non-holonomic frames of second-order of M. Moreover the restriction of the former projection still defines a $\mathrm{GL}(n, \mathbb{K})$-principal bundle. Since a linear connection is a horizontal distribution of n-planes invariant under the action of $\mathrm{GL}(n, \mathbb{K})$ it therefore determines a $\mathrm{GL}(n, \mathbb{K})$-reduction of the bundle $\hat{\mathcal{F}}^2(M)$, in a bijective way. This is a new proof of a theorem of Libermann.

MSC: 58A20; 58A32; 14M15

Keywords: Holonomic; Jet bundle; Grassmann; Stiefel

1. Introduction

Let M be a real or complex differentiable manifold with $\dim_{\mathbb{K}}(M) = n(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$. Jets and jet bundles on M were first defined and studied by Ehresmann [4–8]. Let us recall

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that a holonomic frame u of order r in $x \in M$ (or simply, a frame of order r) is essentially the Taylor polynomial of order r of a $\mathbb{K}$-diffeomorphism $f : (\mathbb{K}^n, 0) \to (M, x)$ (these notations mean that f is bi-holomorphic if $\mathbb{K} = \mathbb{C}$ and that f maps an open neighborhood of $0 \in \mathbb{K}^n$ onto an open neighborhood of $x \in M$, with $f(0) = x$). In other words, u is the jet $j^r_{0,x} f$. Holonomic frames of order r form a principal bundle $F^r(M) = \mathcal{F}^r(M)$ over M with structure group $G^r(n, \mathbb{K})$. The elements of the group are the jets $j^r_{0,0}h$ with $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ a $\mathbb{K}$-diffeomorphism. Group law is jet composition and the identity element is $e_r = j^r_{0,0}(\mathrm{id}_{\mathbb{K}^n})$. The group $G^r(n, \mathbb{K})$ is complex if $\mathbb{K} = \mathbb{C}$ and the bundle $F^r(M)$ is complex if M is a complex manifold. If $r = 1$ then $G^1(n, \mathbb{K})$ is just the general linear group $\mathrm{GL}(n, \mathbb{K})$ and $F^1(M) = F(M) = \mathcal{F}(M) =$ the bundle of $\mathbb{K}$-linear frames on M.

The bundles $\tilde{\mathcal{F}}^r(M) = \tilde{F}^r(M)$ and $\hat{\mathcal{F}}^r(M) = \hat{F}^r(M)$ of non-holonomic and semi-holonomic frames of any order $r > 1$ were introduced by Ehresmann [9,10]. Among the contributors who studied and developed their properties we must cite Libermann [24,25] and Yuen [31]. It is customary to omit the word *holonomic* when one only deals with the smaller bundle $F^r(M)$ (see, for example, [17,20,29]).

Second-order G-structures of non-holonomic, semi-holonomic and holonomic type (that is to say, G-reductions of the respective frame bundle) are important in differential geometry and continuum mechanics. Let us merely point out that non-holonomic second-order G-structures have been applied to continuum mechanics by Epstein and de León to develop the study of generalized Cosserat media and generalized liquid crystals in [12,13,21,22]. On the other hand, a theorem of Libermann [24] states that every linear connection on a real manifold can be characterized as a $\mathrm{GL}(n, \mathbb{R})$-reduction of the bundle of semi-holonomic second-order frames on M, extending a previous result of Kobayashi for torsionless connections [16–18]. Second-order (holonomic) G-structures associated to real or complex semisimple graded Lie algebras were studied by Ochiai [26,27], generalizing the previous theory of projective and conformal structures of Weyl and Cartan, which are holonomic second-order G-structures, as it is shown in Chapter IV of Kobayashi's book [17]. A different class of second-order (holonomic) G-structures on real contact manifolds was found by Burdet and Perrin [3].

Our first main result is Theorem 8.2. It says that the principal bundles $\tilde{F}^2(M)$ and $\hat{F}^2(M)$ of non-holonomic and semi-holonomic frames of second-order on M are, respectively, isomorphic to the bundles $F^2(M)^{\tilde{G}^2(n,\mathbb{K})}$ and $F^2(M)^{\hat{G}^2(n,\mathbb{K})}$ obtained by extending the group $G^2(n, \mathbb{K})$ of the bundle $F^2(M)$ of all holonomic second-order frames of M to the corresponding larger structure groups $\tilde{G}^2(n, \mathbb{K})$ and $\hat{G}^2(n, \mathbb{K})$ of those bundles. It characterizes them in a simple way, allowing us to extend algebraically any local trivialization of $F^2(M)$ to both. This fact allows us to outline a new proof of Libermann's theorem, at the end of this paper.

According to Ehresmann [9], non-holonomic second-order frames are 1-jets of (local) sections of M into $\mathcal{F}(M)$. Yuen [31] considered second-order non-holonomic frames on a real manifold M as 1-jets of bundle isomorphisms of $(\mathcal{F}(\mathbb{R}^n), e_1)$ in $(\mathcal{F}(M), z)$ (see also [12]). Their respective constructions lead to principal bundles over M which are isomorphic. Complex data lead to holomorphic bundles. On the other hand, each semi-holonomic frame has associated an n-dimensional horizontal space [31, p. 13], a fact further studied by de León and Ortacgil [23], who proved that a semi-holonomic frame of second-order on a real manifold M can be considered as an element of $\mathcal{F}(\mathcal{F}M)$ defined by its associated *horizontal space*. In this way, they constructed a new $\hat{G}^2(n, \mathbb{K})$-principal bundle $\hat{H}^2(M) \to M$ which is a regular submanifold of $\mathcal{F}(\mathcal{F}M)$, proving that it is isomorphic to the bundle $\hat{\mathcal{F}}^2(M) \to M$ of semi-holonomic frames of second-order on M. We have observed that their bundle $\hat{H}^2(M)$ is holomorphic if M is complex. We have also adapted their idea to non-holonomic frames of second-order, obtaining them as elements of $\mathcal{F}(\mathcal{F}M)$ defined by *bases of horizontal spaces* on a real or complex manifold M. In this way, we have constructed a new bundle $\tilde{H}^2(M)$ on M that is a regular submanifold of $\mathcal{F}(\mathcal{F}M)$, and a natural projection $\pi_H : \tilde{H}^2(M) \to \hat{H}^2(M)$ which is a $\mathrm{GL}(n, \mathbb{K})$-principal bundle isomorphic to $\tilde{\mathcal{F}}^2(M) \to \hat{\mathcal{F}}^2(M)$ (Theorems 4.1 and 5.2). Bearing again in mind the same idea of using horizontal n-bases and horizontal n-planes in $\mathcal{F}(M)$, we will obtain other new models for non-holonomic and semi-holonomic frames of second-order of M as open sub-bundles of what we have called the *Stiefel and Grassmann tangent bundles* of $\mathcal{F}(M)$. This construction is based on a geometric approach to coordinates in Grassmann manifolds (considered as $\mathrm{GL}(q, \mathbb{K})$-homogeneous spaces) which, as far as we know, is due to Hangan [15]. We will give it in a way slightly different to his, and then we will associate to the tangent bundle $E = T(N)$ of an arbitrary real or complex manifold N the *Stiefel tangent bundles* of N and the *Grassmann tangent bundles* of N. For each $0 < q < \dim_{\mathbb{K}} N$ they are, respectively, defined as follows:

$$\mathcal{S}t_q(E) = \bigcup_{x\in N} \mathcal{S}t_q(E_x), \qquad \mathcal{G}_q(E) = \bigcup_{x\in N} \mathcal{G}_q(E_x),$$

where $E_x = T_xN$ is the fiber on $x \in N$. Given a real or complex manifold M we will just need the particular case $q = n = \dim_{\mathbb{K}}(M)$, $N = \mathcal{F}(M)$ and therefore $\dim_{\mathbb{K}}(N) = n + n^2$. Let $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M}$ be the set of all n-bases of all horizontal tangent n-planes at the points of $\mathcal{F}(M)$ and $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$ the set of all horizontal tangent n-planes at the points of $\mathcal{F}(M)$. We will prove (Theorem 3.1) that $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to M$ and $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M} \to M$ are open sub-bundles of $\mathcal{S}t_n(E) \to M$ and $\mathcal{G}_n(E) \to M$, respectively. Moreover, the natural projection $\pi_{\mathcal{H}}$: $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to \mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$ is also a $\mathrm{GL}(n, \mathbb{K})$-principal bundle. It induces a naturally isomorphic $\mathrm{GL}(n, \mathbb{K})$-principal bundle structure $\tilde{\mathcal{F}}^2(M) \to \hat{\mathcal{F}}^2(M)$ (Theorem 5.2). These facts and Theorem 6.1 yield isomorphisms between the bundle $\tilde{\mathcal{F}}^2(M) \to M$ of non-holonomic frames of second-order on M and any of the bundles $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to M$, $\tilde{H}^2(M) \to M$, whereas the bundle $\hat{\mathcal{F}}^2(M) \to M$ of semi-holonomic frames of second-order on M is isomorphic to any of the bundles $\hat{H}^2(M) \to M$ and $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M} \to M$ (Theorem 6.2). Using our bundle $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M} \to M$ of horizontal n-planes we will easily obtain that a (holomorphic) linear connection on a real (complex) manifold is a $\mathrm{GL}(n, \mathbb{K})$-reduction of the isomorphic bundle $\hat{\mathcal{F}}^2(M) \to M$ of semi-holonomic frames of second-order on M (Theorem 7.1), a result of Libermann quoted before, which will be also deduced using Theorem 8.2.

A warning about notations: for a complex manifold M, its tangent bundle $T(M)$ is a holomorphic vector bundle and the $\mathrm{GL}(n, \mathbb{C})$-principal bundle denoted by $F(M) = \mathcal{F}(M)$ contains only $\mathbb{C}$-linear frames and it is strictly contained in the the $\mathrm{GL}(2n, \mathbb{R})$-principal bundle of $\mathbb{R}$-linear frames on the underlying real manifold $M_{\mathbb{R}} = M$. Analogously for any kind of higher order frames. The properties that we have dealt with are formally almost identical for real or complex data. But the complex case deserves a mention, mainly because holomorphic connections do not always exist, [1]. It is clear how to adapt things when data are real analytic instead of holomorphic.

We thank Professor J.A. Oubiña for his technical advice and to Professor M. de León for some useful conversations on the subject. His bundle $H^2(M) \subset \mathcal{F}(\mathcal{F}M)$ in [23] has been denoted $\hat{H}^2(M)$ by us.

2. Stiefel and Grassmann tangent bundles over a manifold

For the sake of completeness, we shall first review what Grassmann and Stiefel manifolds are. We will need them in our treatment of second-order non-holonomic and semi-holonomic frame bundles on a real or complex manifold.

Let $p, q \in \mathbb{N}$, $1 \leq p, q \leq m$, $p+q = m$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{M}_{m\times q}(\mathbb{K}) = m \times q$ matrices with components in $\mathbb{K}$. A q-basis of $\mathbb{K}^m$ (or q-reference of $\mathbb{K}^m$, not to be confused with a reference of kth order) is an element of $\mathbb{K}^m \times \cdots^{q)} \times \mathbb{K}^m$ with linearly independent vector components. In an equivalent way, a q-basis is an injective $\mathbb{K}$-linear map from $\mathbb{K}^q$ into $\mathbb{K}^m$. The Stiefel manifold $\mathcal{S}t_q(\mathbb{K}^m)$ is the set of all the q-bases of $\mathbb{K}^m$. It is an open set in $\mathbb{K}^m \times \cdots^{q)} \times \mathbb{K}^m$. If we fix a basis $\mathcal{B} = \{e_1, \ldots, e_p, e_{p+1}, \ldots, e_m\}$ of $\mathbb{K}^m$ then any q-basis $\{w_{p+1}, \ldots, w_m\}$ of $\mathbb{K}^m$ can be expressed in the form $w_j = n^i_j e_i$, $j = p+1, \ldots, m$. Thus, $\mathcal{B}$ defines a global chart in $\mathcal{S}t_q(\mathbb{K}^m)$ as follows: If a q-dimensional subspace W is generated by a q-reference $\mathcal{B}_W = \{w_{p+1}, \ldots, w_m\}$ then we send $\mathcal{B}_W$ to the $m \times q$ matrix $N_{\mathcal{B}_W} = [n^i_j]$, which has rank q, and we will write $W = \langle \mathcal{B}_W \rangle$. Let us recall that two q-references are considered equivalent if they generate the same q-dimensional vector subspace (or q-plane, for short). The quotient space $\mathcal{G}_q(\mathbb{K}^m)$ is the Grassmann manifold and, therefore, it is the set of all q-planes of $\mathbb{K}^m$ with the quotient topology. It is an analytical real or complex manifold of real or complex dimension pq. Coordinates on $\mathcal{G}_q(\mathbb{K}^m)$ are classically given as follows: Let us denote as σ an increasing sequence $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_q \leq m$ of $q = m - p$ integer numbers. Associated to σ we can define the subset $A_{(\sigma,\mathcal{B})}$ of $\mathcal{G}_q(\mathbb{K}^m)$, formed by all the q-planes $W = \langle \mathcal{B}_W \rangle$ such that the rows $\sigma_1, \sigma_2, \ldots, \sigma_q$ of the previous matrix $N_{\mathcal{B}_W}$ are linearly independent. This subset $A_{(\sigma,\mathcal{B})}$ is an open set in the quotient topology of $\mathcal{G}_q(\mathbb{K}^m)$. Furthermore, if $W = \langle \mathcal{B}_W \rangle = \langle \{w_{p+1}, \ldots, w_m\} \rangle \in A_{(\sigma,\mathcal{B})}$, then the matrix $N^{\sigma}_{\mathcal{B}_W} = [n^{\sigma_i}_j] \in \mathcal{M}_{q\times q}(\mathbb{K})$, whose ith row is the σ_ith row of $N_{\mathcal{B}_W}$, is non-singular. The matrix product $N_{\mathcal{B}_W}(N^{\sigma}_{\mathcal{B}_W})^{-1}$ is an $m \times q$-matrix whose σ_ith row coincides with the ith row of the identity matrix $I \in \mathcal{M}_{q\times q}(\mathbb{K})$, where $i = 1, \ldots, q$. The remaining p rows in $N_{\mathcal{B}_W}(N^{\sigma}_{\mathcal{B}_W})^{-1}$ form (by definition) the $p \times q$ real or complex matrix X of coordinates of the q-plane W, that can be denoted as $X = \Phi_{(\sigma,\mathcal{B})}(W)$, since they depend both on the sequence σ and on the basis $\mathcal{B}$ chosen in $\mathbb{K}^m$; notice, however, that X does not depend on the q-basis $\mathcal{B}_W$ chosen in W. If Σ denotes the set of increasing sequences $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_q \leq m$, it is well known that the family $\cup_{\sigma,\mathcal{B}}\{(A_{(\sigma,\mathcal{B})}, \Phi_{(\sigma,\mathcal{B})})\}$ (where $\sigma \in \Sigma$ and $\mathcal{B}$ runs through all bases of $\mathbb{K}^m$) is an analytic atlas, since the more general change of coordinates is given by

$$X \to (AX + B)(CX + D)^{-1}$$

for suitable matrices A, B, C and D (see [15] for details). It is just the atlas that $\mathcal{G}_q(\mathbb{K}^m)$ has as a homogeneous space under the natural left action of $\mathrm{GL}(n, \mathbb{K})$. The natural projection

$\pi : \mathcal{S}t_q(\mathbb{K}^m) \to \mathcal{G}_q(\mathbb{K}^m)$ sending a q-basis $\mathcal{B}_W$ to the q-plane $W = \langle \mathcal{B}_W \rangle$ (i.e., sending each injective linear map $f : \mathbb{K}^q \to \mathbb{K}^m$ to its image $W = f(\mathbb{K}^q)$) defines a structure of $\mathrm{GL}(q, \mathbb{K})$-principal bundle. In coordinates, the action of $\mathrm{GL}(q, \mathbb{K})$ on $\mathcal{S}t_q(\mathbb{K}^m)$ is expressed with the matrix product:

$$(\mathcal{B}_W, L) = \left(\begin{bmatrix} \beta \\ \alpha \end{bmatrix}, L \right) \to \mathcal{B}_W L = \begin{bmatrix} \beta L \\ \alpha L \end{bmatrix},$$

where we have identified a q-basis $\mathcal{B}_W$ with the $m \times q$ matrix $N_{\mathcal{B}_W}$ of coordinates of their column vectors in a fixed basis $\mathcal{B}$ of $\mathbb{K}^m$, β and α being, respectively, the submatrices formed by the first p rows and the last q rows. If we consider elements in $\mathcal{S}t_q(\mathbb{K}^m)$ as injective linear maps from $\mathbb{K}^q$ to $\mathbb{K}^m$, then the action of $\mathrm{GL}(q, \mathbb{K})$ on $\mathcal{S}t_q(\mathbb{K}^m)$ is just map composition.

Now we will detail Hangan's equivalent construction of the previous atlas of $\mathcal{G}_q(\mathbb{K}^m)$ in a way slightly different to his (see [15] and also [30]). We will use it in the next section. Let U be a vector subspace of $\mathbb{K}^m$ of dimension $p = m - q$ and let $\mathcal{B} = \{e_1, \dots, e_p, e_{p+1}, \dots, e_m\}$ be a basis of $\mathbb{K}^m$ adapted to U, that is to say, $U = \langle \{e_1, \dots, e_p\} \rangle$. We will denote as S_U the set of all q-planes of $\mathbb{K}^m$ which are supplementary with respect to U

$$S_U = \{W \in \mathcal{G}_q(\mathbb{K}^m); U \oplus W = \mathbb{K}^m\}.$$

It is easy to see that $\pi^{-1}(S_U) = \{\mathcal{B}_W; W \in \mathcal{S}_U\}$ is open in $\mathcal{S}t_q(\mathbb{K}^m)$ and therefore S_U is an open subset in $\mathcal{G}_q(\mathbb{K}^m)$. Let us remark that, for all $W \in S_U$, we have immediately that:

(i) given $v \in \langle \{e_{p+1}, \dots, e_m\} \rangle$, there is a unique $u \in U$ such that $v + u \in W$;
(ii) there exist unique $u_{p+1}, \dots, u_m \in U$ such that $\{e_{p+1} + u_{p+1}, \dots, e_m + u_m\}$ is a basis of W (in an equivalent way, there is a unique basis $\{w_{p+1}, \dots, w_m\}$ of W such that for all $i = p + 1, \dots, m$ is $w_i = e_i + u_i$ with $u_i \in U$).

Proposition 2.1. *The open subset S_U of the Grassmann manifold $\mathcal{G}_q(\mathbb{K}^m)$ is homeomorphic to the pq-dimensional vector subspace $U \times \cdots^{q)} \times U$* of $\mathbb{K}^m \times \cdots^{q)} \times \mathbb{K}^m$.

Proof. Let us define $\Psi : U \times \cdots^{q)} \times U \to S_U$ by

$$\Psi(u_{p+1}, \dots, u_m) = \langle \{e_{p+1} + u_{p+1}, \dots, e_m + u_m\} \rangle.$$

It follows from the previous remark that the map Ψ is surjective. On the other hand, if $u_{p+1}, \dots, u_m, \tilde{u}_{p+1}, \dots, \tilde{u}_m \in U$ satisfy

$$\langle \{e_{p+1} + u_{p+1}, \dots, e_m + u_m\} \rangle = \langle \{e_{p+1} + \tilde{u}_{p+1}, \dots, e_m + \tilde{u}_m\} \rangle,$$

then, for every $i \in \{p + 1, \dots, m\}$, it is

$$e_i + u_i = \alpha_{p+1}(e_{p+1} + \tilde{u}_{p+1}) + \cdots + \alpha_m(e_m + \tilde{u}_m).$$

Using that $\mathbb{K}^m = U \oplus \langle \{e_{p+1}, \dots, e_m\} \rangle$ we get

$$e_i = \alpha_{p+1} e_{p+1} + \cdots + \alpha_m e_m, \qquad u_i = \alpha_{p+1} \tilde{u}_{p+1} + \cdots + \alpha_m \tilde{u}_m.$$

Since $e_{p+1}, \dots, e_m$ are linearly independent, we must have $\alpha_i = 1$ and $\alpha_j = 0$ if $j \neq i$; therefore

$$u_i = \tilde{u}_i.$$

Thus, the map Ψ is injective too. It is immediate that both Ψ and Ψ^{-1} are continuous. □

Definition 2.2. Let $W \in S_U$, $W = \Psi(u_{p+1}, \dots, u_m)$. If $u_j = m^i_j e_i$, $j = p+1, \dots, m$, we will say that the matrix

$$M_W = [m^i_j] \in \mathcal{M}_{p\times q}(\mathbb{K})$$

is the matrix of coordinates of W.

Clearly, M_W depends on the chosen basis $\mathcal{B}$ adapted to U. It is also clear that

Proposition 2.3. *Let U be a p-dimensional vector subspace of* $\mathbb{K}^m$ *and let* $\mathcal{B}$ *be a basis of* $\mathbb{K}^m$ *adapted to U. The mapping* $\Phi_{(U,\mathcal{B})} : S_U \to \mathcal{M}_{p\times q}(\mathbb{K})$ *defined by*

$$\Phi_{(U,\mathcal{B})}(W) = M_W$$

is a chart in $\mathcal{G}_q(\mathbb{K}^m)$*. We get the previous atlas of* $\mathcal{G}_q(\mathbb{K}^m)$ *varying U and* $\mathcal{B}$ *in all possible ways.*

Let us fix a p-dimensional vector subspace U of $\mathbb{K}^m$ and a basis $\mathcal{B} = \{e_1, \dots, e_p, e_{p+1}, \dots, e_m\}$ of $\mathbb{K}^m$ adapted to U. If $\{w_{p+1}, \dots, w_m\}$ is a basis of $W = \Psi(u_{p+1}, \dots, u_m) \in S_U$, with

$$w_j = \sum_{i=1}^{p} \beta^i_j e_i + \sum_{i=p+1}^{m} \alpha^i_j e_i, \quad j = p+1, \dots, m,$$

we will denote $\alpha = [\alpha^i_j] \in \mathcal{M}_{q\times q}(\mathbb{K})$, $\beta = [\beta^i_j] \in \mathcal{M}_{p\times q}(\mathbb{K})$. It is immediate that the numbers β^i_j and α^i_j are unique and the matrix $\alpha = [\alpha^i_j]$ is non-singular. Moreover, we have

Lemma 2.4. *The matrix* M_W *of coordinates of the q-plane W is*

$$M_W = \beta\alpha^{-1}.$$

Remark 2.5. Conversely, fixed a basis $\mathcal{B} = \{e_1, \dots, e_p, e_{p+1}, \dots, e_m\}$ of $\mathbb{K}^m$ adapted to U, if $\beta = [\beta^i_j] \in \mathcal{M}_{p\times q}(\mathbb{K})$ and $\alpha = [\alpha^i_j] \in \mathcal{M}_{q\times q}(\mathbb{K})$ is non-singular, then, defining w_j, $j = p+1, \dots, m$, as above, it is clear that $\mathcal{B}_W = \{w_{p+1}, \dots, w_m\}$ is a q-basis generating a q-plane $W \in S_U$ with coordinates $M_W = \beta\alpha^{-1}$. Additionally, the map sending $\mathcal{B}_W$ to

$$\begin{bmatrix} \beta \\ \alpha \end{bmatrix} \in \mathcal{M}_{m\times q}(\mathbb{K})$$

is a diffeomorphism of $\pi^{-1}(S_U)$ onto an open subset of $\mathcal{M}_{m\times q}(\mathbb{K})$.

Let now M be a real or complex manifold of dimension m and $\pi_E : E \to M$ be a vector bundle of fiber $F = \mathbb{K}^r$. Given $0 < q < r$, we define the bundles

$$\mathcal{S}t_q(E) = \bigcup_{x\in M} \mathcal{S}t_q(\pi_E^{-1}(x)), \qquad \mathcal{G}_q(E) = \bigcup_{x\in M} \mathcal{G}_q(\pi_E^{-1}(x))$$

and the projections

$$\pi_{\mathcal{S}t} : \mathcal{B}_{W_x} \in \mathcal{S}t_q(E) \to x \in M, \qquad \pi_{\mathcal{G}} : W_x \in \mathcal{G}_q(E) \to x \in M,$$

where $W_x = \langle\{w_1, \ldots, w_q\}\rangle \in \mathcal{G}_q(\pi_E^{-1}(x))$ is generated by the q-reference $\mathcal{B}_{W_x} = \{w_1, \ldots, w_q\}$. Let $\tau : \pi_E^{-1}(A) \to A \times F$ be a local trivialization of E and $p : (x, v) \in A \times F \to v \in F$. Then

$$\pi_{\mathcal{S}t}^{-1}(A) = \mathcal{S}t_k(E)_{/A} = \bigcup_{x\in A} \mathcal{S}t_q(\pi_E^{-1}(x)),$$
$$\pi_{\mathcal{G}}^{-1}(A) = \mathcal{G}_k(E)_{/A} = \bigcup_{x\in A} \mathcal{G}_q(\pi_E^{-1}(x)),$$

and the maps defined by

$$\mathcal{S}t_q(\tau) : \mathcal{B}_{W_x} \in \pi_{\mathcal{S}t}^{-1}(A) \to (x, \{p\circ\tau(w_1), \ldots, p\circ\tau(w_q)\}) \in A \times \mathcal{S}t_q(F),$$
$$\mathcal{G}_q(\tau) : W_x \in \pi_{\mathcal{G}}^{-1}(A) \to (x, \langle\{p\circ\tau(w_1), \ldots, p\circ\tau(w_1)\}\rangle) \in A \times \mathcal{G}_q(F)$$

are local trivializations of $\mathcal{S}t_q(E)$ and $\mathcal{G}_q(E)$. Moreover, $\mathrm{GL}(q, \mathbb{K})$ acts freely on $\mathcal{S}t_q(E)$ as follows: if $u : \mathbb{K}^q \to E_x \equiv \mathbb{K}^m$ is a q-reference and $a \in \mathrm{GL}(q, \mathbb{K})$, then, $ua = u \circ a$. The map

$$\pi_{\mathcal{G}}^{\mathcal{S}t} : \mathcal{B}_{W_x} \in \mathcal{S}t_q(E) \to W_x \in \mathcal{G}_q(E)$$

defines a structure of $\mathrm{GL}(q, \mathbb{K})$-principal bundle. It is holomorphic if $\mathbb{K} = \mathbb{C}$, M is complex and $\pi_E : E \to M$ is holomorphic.

Definition 2.6. If $E = TM$, we will say that $\mathcal{S}t_q(TM)$ and $\mathcal{G}_q(TM)$ are, respectively, the Stiefel and Grassmann tangent bundles of q-references and q-planes over M, $0 < q < m = \dim_{\mathbb{K}}(M)$.

Since the group $\mathrm{GL}(m, \mathbb{K})$ acts on the right on $\mathcal{F}(M)$ and on the left on both $\mathcal{S}t_q(\mathbb{K}^m)$ and $\mathcal{G}_q(\mathbb{K}^m)$, we can consider the associated bundles $\mathcal{F}(M) \times_{\mathrm{GL}(m,\mathbb{K})} \mathcal{S}t_q(\mathbb{K}^m)$ of fiber $\mathcal{S}t_q(\mathbb{K}^m)$ and $\mathcal{F}(M) \times_{\mathrm{GL}(m,\mathbb{K})} \mathcal{G}_q(\mathbb{K}^m)$ of fiber $\mathcal{G}_q(\mathbb{K}^m)$, induced by those actions. They are, respectively, isomorphic to $\mathcal{S}t_q(TM)$ and $\mathcal{G}_q(TM)$. Let us point out that $\mathcal{G}_q(TM)$ is denoted as $\mathcal{G}_q(M)$ in [28].

3. Horizontal spaces and bases of horizontal spaces

Now we will introduce the open submanifold $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \subset \mathcal{S}t_n(T(\mathcal{F}M))$ of horizontal n-bases on $\mathcal{F}(M)$ and the open submanifold $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M} \subset \mathcal{G}_n(T(\mathcal{F}M))$ of horizontal n-planes on $\mathcal{F}(M)$. We will later prove that they are in fact principal bundles over M, respectively isomorphic to the bundles of non-holonomic and semi-holonomic second-order frames on M. Each (holomorphic) chart $\phi : x \in A \to [x^i] \in \phi(A)$, defined on an open subset A of a real (complex) manifold M, induces a (holomorphic) trivialization chart $z \in (\pi_0^1)^{-1}(A) \to (x^i, x^i_j) \in \phi(A) \times \mathrm{GL}(n, \mathbb{K})$ on the bundle $\pi_0^1 : \mathcal{F}(M) \to M$ of $\mathbb{K}$-linear frames on M, satisfying

$$\pi_0^1(z) = x \equiv [x^i], \qquad z(E_j) = x^i_j \frac{\partial}{\partial x^i}(x),$$

where $\{E_1, \ldots, E_n\}$ is the usual basis of $\mathbb{K}^n$. The vector space $T_z(\mathcal{F}M)$ is isomorphic to $\mathbb{K}^n \oplus \mathfrak{gl}(n, \mathbb{K})$, where $\mathfrak{gl}(n, \mathbb{K}) = \mathcal{M}_{n\times n}(\mathbb{K}) = \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^n)$ is the Lie algebra of $\mathrm{GL}(n, \mathbb{K})$. As in Section 2, we have two associated bundles

$$\mathcal{S}t_n(T(\mathcal{F}M)) = \bigcup_{z\in\mathcal{F}(M)} \mathcal{S}t_n(T_z(\mathcal{F}M)), \qquad \mathcal{G}_n(T(\mathcal{F}M)) = \bigcup_{z\in\mathcal{F}(M)} \mathcal{G}_n(T_z(\mathcal{F}M)),$$

where $\mathcal{S}t_n(T_z(\mathcal{F}M))$ and $\mathcal{G}_n(T_z(\mathcal{F}M))$ denote, respectively, the Stiefel manifold of n-references of $T_z(\mathcal{F}M)$ and the Grassmann manifold of all n-planes in $T_z(\mathcal{F}M)$. The natural projection

$$\pi_{\mathcal{G}}^{\mathcal{S}t} : \mathcal{B}_W \in \mathcal{S}t_n(T(\mathcal{F}M)) \to W \in \mathcal{G}_n(T(\mathcal{F}M))$$

determines a structure of $\mathrm{GL}(n, \mathbb{K})$-principal bundle. Let us fix in $T_z(\mathcal{F}M)$ the basis

$$\mathcal{B} = \left\{ \frac{\partial}{\partial x^i_j}(z);\ \ i, j = 1, \ldots, n \right\} \bigcup \left\{ \frac{\partial}{\partial x^k}(z);\ \ k = 1, \ldots, n \right\},$$

and let us choose

$$U = V_z = \left\langle \left\{ \frac{\partial}{\partial x^i_j}(z);\ \ i, j = 1, \ldots, n \right\} \right\rangle,$$

the vertical tangent space at $z \in \mathcal{F}(M)$. Bearing in mind previous notations, we have that the basis $\mathcal{B}$ is adapted to U, that $S_U = S_{V_z}$ is the set of horizontal n-spaces at z (with $\dim_{\mathbb{K}}(U) = n^2$) and that $\pi_{\mathcal{G}}^{\mathcal{S}t^{-1}}(S_U)$ is the set of all bases of all horizontal n-spaces at $z \in \mathcal{F}(M)$. Let us define

$$\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} = \bigcup_{z\in\mathcal{F}(M)} \pi_{\mathcal{G}}^{\mathcal{S}t^{-1}}(S_{V_z}), \qquad \mathcal{G}_n\mathcal{H}_{\mathcal{F}M} = \bigcup_{z\in\mathcal{F}(M)} S_{V_z}.$$

If $\mathcal{B}_{H_z} = \{h_1, \ldots, h_n\}$ is a basis of a horizontal n-space $H_z = \langle \mathcal{B}_{H_z} \rangle$ then

$$h_j = \alpha^i_j \frac{\partial}{\partial x^i}(z) + \beta^{k_i}{}_j \frac{\partial}{\partial x^k{}_i}(z), \quad j = 1, \ldots, n,$$

where $\alpha = [\alpha^i_j] \in \mathrm{GL}(n, \mathbb{K})$ and $\beta = [\beta^{k_i}{}_j] \in \mathcal{M}_{n^2\times n}(\mathbb{K})$. Notice that $\mathcal{M}_{n^2\times n}(\mathbb{K})$ is isomorphic to the vector space $\mathrm{Hom}(\mathbb{K}^n, \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^n))$. For $L \in \mathrm{GL}(n, \mathbb{K})$, the horizontal bases $\mathcal{B}_{H_z}$ and $\mathcal{B}_{H_z}L$ (considered as injective $\mathbb{K}$-linear maps) can be identified with matrices

$$\mathcal{B}_{H_z} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \quad \text{and} \quad \mathcal{B}_{H_z} L \equiv \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \equiv L = \begin{bmatrix} \beta L \\ \alpha L \end{bmatrix},$$

giving rise to a free action of $\mathrm{GL}(n, \mathbb{K})$ on $\mathcal{S}t_n \mathcal{H}_{\mathcal{F}M}$.

Theorem 3.1. *The projections $\mathcal{S}t_n \mathcal{H}_{\mathcal{F}M} \to M$ and $\mathcal{G}_n \mathcal{H}_{\mathcal{F}M} \to M$ are, respectively, open sub-bundles of $\pi_{\mathcal{S}t} : \mathcal{S}t_n(T(\mathcal{F}M)) \to M$ and $\pi_{\mathcal{G}} : \mathcal{G}_n(T(\mathcal{F}M)) \to M$. Furthermore, the restriction $\pi_{\mathcal{H}}$ of the natural projection $\pi_{\mathcal{G}}^{\mathcal{S}t} : \mathcal{S}t_n(T(\mathcal{F}M)) \to \mathcal{G}_n(T(\mathcal{F}M))$ determines a* $\mathrm{GL}(n, \mathbb{K})$*-principal bundle*

$$\pi_{\mathcal{H}} : \mathcal{S}t_n \mathcal{H}_{\mathcal{F}(M)} \to \mathcal{G}_n \mathcal{H}_{\mathcal{F}(M)}.$$

Proof. If $\tau : \pi_{T(\mathcal{F}M)}^{-1}(A) \to A \times (\mathbb{K}^n \oplus \mathfrak{gl}(n, \mathbb{K}))$ is the trivialization of $T(\mathcal{F}M)$ induced by a chart (A, ϕ) of M, then the maps

$$\mathcal{S}t_n(\tau) : \mathcal{S}t_n(T(\mathcal{F}M))_{/A} \to (\pi_0^1)^{-1}(A) \times \mathcal{S}t_n(\mathbb{K}^n \oplus \mathfrak{gl}(n, \mathbb{K})),$$
$$\mathcal{G}_n(\tau) : \mathcal{G}_n(T(\mathcal{F}M))_{/A} \to (\pi_0^1)^{-1}(A) \times \mathcal{G}_n(\mathbb{K}^n \oplus \mathfrak{gl}(n, \mathbb{K})),$$

defined as in Section 2, are trivializations of $\mathcal{S}t_n(T(\mathcal{F}M))$ and $\mathcal{G}_n(T(\mathcal{F}M))$. Moreover, if we now call $U = \mathfrak{gl}(n, \mathbb{K})$, and $\pi : \mathcal{S}t_n(\mathbb{K}^n \oplus \mathfrak{gl}(n, \mathbb{K})) \to \mathcal{G}_n(\mathbb{K}^n \oplus \mathfrak{gl}(n, \mathbb{K}))$ is the usual projection, then

$$\mathcal{S}t_n(\tau)(\mathcal{S}t_n \mathcal{H}_{\mathcal{F}(M)_{/A}}) = (\pi_0^1)^{-1}(A) \times \pi^{-1}(S_U),$$
$$\mathcal{G}_n(\tau)(\mathcal{G}_n \mathcal{H}_{\mathcal{F}(M)_{/A}}) = (\pi_0^1)^{-1}(A) \times S_U.$$

The first assertion of the theorem then follows from this and from Propositions 2.1 and 2.3. The second one is true because bases of horizontal n-spaces are horizontal and hence

$$\pi_{\mathcal{G}}^{\mathcal{S}t^{-1}}(\mathcal{G}_n \mathcal{H}_{\mathcal{F}(M)}) = \mathcal{S}t_n \mathcal{H}_{\mathcal{F}(M)}.$$

Furthermore, we have seen that the right action of $\mathrm{GL}(n, \mathbb{K})$ on $\mathcal{S}t_n(T(\mathcal{F}M))$ preserves $\mathcal{S}t_n \mathcal{H}_{\mathcal{F}M}$. Therefore, it determines a $\mathrm{GL}(n, \mathbb{K})$-principal bundle $\pi_{\mathcal{H}} : \mathcal{S}t_n \mathcal{H}_{\mathcal{F}(M)} \to \mathcal{G}_n \mathcal{H}_{\mathcal{F}(M)}$. In these trivializations, the coordinates of the horizontal n-basis $\mathcal{B}_{H_z} = \{h_1, \dots, h_n\}$ are

$$(x^i, x^i_j, \alpha^i_j, \beta^{ij}{}_k) = ([x^i], X, \alpha, \beta) \in \phi(A) \times \mathrm{GL}(n, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K}) \times \mathcal{M}_{n^2 \times n}(\mathbb{K}),$$

whereas the coordinates of the horizontal n-plane $H_z = \langle \mathcal{B}_{H_z} \rangle = \langle \{h_1, \dots, h_n\} \rangle$ are

$$([x^i], X, \beta\alpha^{-1}) \in \phi(A) \times \mathrm{GL}(n, \mathbb{K}) \times \mathcal{M}_{n^2 \times n}(\mathbb{K}).$$

Notice that, in the chosen basis $\mathcal{B}$ of $T_z(\mathcal{F}M)$, the matrix M_{H_z} of coordinates of H_z in the Grassmann manifold $\mathcal{G}_n(T_z(\mathcal{F}M))$ is, by virtue of Lemma 2.4, just $M_{H_z} = \beta\alpha^{-1} \in \mathcal{M}_{n^2 \times n}(\mathbb{K})$. The projection $\pi_{\mathcal{H}}$ is expressed in coordinates as

$$([x^i], X, \alpha, \beta) \to ([x^i], X, \beta\alpha^{-1}). \qquad \square$$

4. Linear frames on $\mathcal{F}(M)$ defined by horizontal bases and by horizontal spaces

In this section we will express the bundles $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to M$ and $\mathcal{G}_n\mathcal{H}_{\mathcal{F}(M)} \to M$ in terms of $\mathbb{K}$-linear frames on the manifold $\mathcal{F}(M)$. Moreover, we will add to the manifold $\hat{H}^2(M)$ ($=H^2(M)$ in [23]) a new manifold $\tilde{H}^2(M)$ projecting over $\hat{H}^2(M)$ in such a way that $\tilde{H}^2(M) \to \hat{H}^2(M)$ will become a $\mathrm{GL}(n,\mathbb{K})$-principal bundle isomorphic to $\pi_{\mathcal{H}} : \mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to \mathcal{G}_n\mathcal{H}_{\mathcal{F}(M)}$. Let $H_z \subset T_z(\mathcal{F}M)$ be a horizontal n-space at $z \in \mathcal{F}(M)$ and let $\mathcal{B}_{H_z} = \{h_1, \dots, h_n\}$ be an arbitrary basis of H_z. We will now associate to $\mathcal{B}_{H_z}$ a $\mathbb{K}$-linear frame $\tilde{u}(\mathcal{B}_{H_z})$ given by

$$\tilde{u}(\mathcal{B}_{H_z}) : (E_j, Y) \in \mathbb{K}^n \oplus \mathfrak{gl}(n,\mathbb{K}) \to h_j + Y^*_z \in T_z(\mathcal{F}M),$$

where $\{E_j\}$ is the usual basis of $\mathbb{K}^n$ and Y^* the fundamental vector field corresponding to $Y = [Y^i_j] \in \mathfrak{gl}(n,\mathbb{K})$. Let (x^i, x^i_j) be the coordinates of $z \in \mathcal{F}(M)$ and $\{E^i_j\}$ the usual basis of $\mathfrak{gl}(n,\mathbb{K})$. If

$$\tilde{u}(\mathcal{B}_{H_z})(E_j) = \alpha^i_j \frac{\partial}{\partial x^i}(z) + \beta^{k_i}{}_j \frac{\partial}{\partial x^k{}_i}(z),$$
$$\tilde{u}(\mathcal{B}_{H_z})(E^i_j) = \gamma^k{}_{i_j} \frac{\partial}{\partial x^k{}_i}(z) + \epsilon^{k_l}{}_{i_j} \frac{\partial}{\partial x^k{}_l}(z),$$

then it follows that all $\gamma^k{}_{i_j} = 0$, since $\tilde{u}(\mathcal{B}_{H_z})(E^i_j) = (E^i_j)^*_z$ is vertical. Moreover,

$$Y^*_z = (x^i_s Y^s_j) \frac{\partial}{\partial x^i_j}(z)$$

and therefore

$$\epsilon^{k_l}{}_{i_j} = x^k{}_i \delta^l{}_j.$$

Let $\tilde{H}^2(M) \subset \mathcal{F}(\mathcal{F}M)$ be the set of all frames over $\mathcal{F}(M)$ defined by n-bases of horizontal n-planes in the way described before, that is to say,

$$\tilde{H}^2(M) = \{\tilde{u}(\mathcal{B}_{H_z}); \mathcal{B}_{H_z} \in \mathcal{S}t_n\mathcal{H}_{\mathcal{F}M}\}.$$

An element in $\mathcal{F}(\mathcal{F}M)$ with coordinates $(x^i, x^i_j, \alpha^i_j, \beta^{h_i}{}_j, \gamma^h{}_{i_j}, \epsilon^{h_k}{}_{i_j})$ belongs to $\tilde{H}^2(M)$ if and only if $(x^i, x^i_j, \alpha^i_j, \beta^{i_j}{}_k, \gamma^i{}_{j_k}, \epsilon^{h_k}{}_{i_j}) = (x^i, x^i_j, \alpha^i_j, \beta^{i_j}{}_k, 0, x^h{}_i \delta^k{}_j)$. In particular, associated with each chart (A, ϕ) on M, there is a local coordinate system in $\tilde{H}^2(M)$ of the form $(x^i, x^i_j, \alpha^i_j, \beta^{i_j}{}_k)$. This shows that $\tilde{H}^2(M)$ is a regular submanifold of $\mathcal{F}(\mathcal{F}M)$.

Let us recall that the canonical form θ of $\mathcal{F}(M)$ sends $v \in T_z(\mathcal{F}M)$ to $\theta(v) = z^{-1}(\pi^1_{0*}(z) \cdot v) \in \mathbb{K}^n$. If $H_z \subset T_z(\mathcal{F}M)$ is a horizontal n-plane, the restriction $\theta_{/H_z} : H_z \to \mathbb{K}^n$ is an isomorphism and $\mathcal{B}_{H_z} = \{\theta^{-1}_{/H_z}(E_1), \dots, \theta^{-1}_{/H_z}(E_n)\}$ is a basis of H_z. Just as it is done in [14,23,31], we can now associate to each horizontal n-plane $H_z \subset T_z(\mathcal{F}M)$ a $\mathbb{K}$-linear frame $\hat{u}(H_z)$ at $z \in \mathcal{F}(M)$, given by

$$\hat{u}(H_z) = \tilde{u}(\mathcal{B}_{H_z}),$$

that is to say,

$$\hat{u}(H_z) : (\xi, Y) \in \mathbb{K}^n \oplus \mathfrak{gl}(n, \mathbb{K}) \to h + Y_z^* \in T_z(\mathcal{F}M),$$

where $h \in H_z$ is determined by $\theta(h) = \xi$. Let $\hat{H}^2(M) \subset \mathcal{F}(\mathcal{F}M)$ be the set defined by

$$\hat{H}^2(M) = \{\hat{u}(H_z); H_z \in \mathcal{G}_n\mathcal{H}_{\mathcal{F}M}\}.$$

We have said that $\hat{H}^2(M)$ was constructed, and denoted $H^2(M)$, in [23]. An element of $\mathcal{F}(\mathcal{F}M)$ with local coordinates $(x^i, x^i_j, \alpha^i_j, \beta^{ij}{}_k, \gamma^i{}_{jk}, \epsilon^{h_k}{}_{ij})$ belongs to $\hat{H}^2(M)$ if and only if

$$(x^i, x^i_j, \alpha^i_j, \beta^{ij}{}_k, \gamma^i{}_{jk}, \epsilon^{h_k}{}_{ij}) = (x^i, x^i_j, x^i_j, \beta^{ij}{}_k, 0, x^h{}_i, \delta^k_j).$$

In particular, associated with each chart (A, ϕ) on M, there is a local coordinate system in $\hat{H}^2(M)$ of the form $(x^i, x^i_j, \beta^{ij}{}_k)$. Hence, $\hat{H}^2(M)$ is a regular submanifold of $\mathcal{F}(\mathcal{F}M)$. As well notice that an element $(x^i, x^i_j, \alpha^i_j, \beta^{ij}{}_k) \in \tilde{H}^2(M)$ belongs to $\hat{H}^2(M)$ if and only if $\alpha^i_j = x^i_j$. It is so because

$$\theta^{-1}_{/H_z}(E_j) = x^i_j \frac{\partial}{\partial x^i}(z) + \beta^{k_l}{}_j \frac{\partial}{\partial x^k{}_l}(z), \quad j = 1, \dots, n.$$

Let $\pi_H : \tilde{H}^2(M) \to \hat{H}^2(M)$ be given by $\pi_H(\tilde{u}(\mathcal{B}_{H_z})) = \hat{u}(H_z)$. If $\tilde{u}(\mathcal{B}_{H_z}) \in \tilde{H}^2(M)$ and

$$\tilde{u}(\mathcal{B}_{H_z})(E_j) = \alpha^i_j \frac{\partial}{\partial x^i}(z) + \beta^{k_l}{}_j \frac{\partial}{\partial x^k{}_l}(z)$$

then, if $L = [L^i_j] \in \mathrm{GL}(n, \mathbb{K})$, we define $\tilde{u}(\mathcal{B}_{H_z})L$ by

$$(\tilde{u}(\mathcal{B}_{H_z})L)(E_j, Y) = \tilde{h}_j + Y_z^*, \quad j = 1 \dots, n,$$

where

$$\tilde{h}_j = (\alpha^i_r L^r_j) \frac{\partial}{\partial x^i}(z) + (\beta^{k_l}{}_s L^s_j) \frac{\partial}{\partial x^k{}_l}(z).$$

Theorem 4.1. *The maps* $\pi_{\mathcal{H}} : \mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to \mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$ *and* $\pi_H : \tilde{H}^2(M) \to \hat{H}^2(M)$ *are isomorphic* $\mathrm{GL}(n, \mathbb{K})$*-principal bundles. In fact,*

$$(\tilde{u}, \hat{u}) : (\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M}, \mathcal{G}_n\mathcal{H}_{\mathcal{F}M}) \to (\tilde{H}^2(M), \hat{H}^2(M))$$

is an isomorphism of principal bundles.

Proof. For every $L = [L^i_j] \in \mathrm{GL}(n, \mathbb{K})$ we have that $\tilde{u}(\mathcal{B}_{H_z}L) = \tilde{u}(\mathcal{B}_{H_z})L$. Furthermore, both $\tilde{u}$ and $\hat{u}$ are diffeomorphisms, because, in local coordinates induced by a chart of M, $\tilde{u}$ is the identity, whereas $\hat{u}$ is given by

$$\hat{u}([x_i], X, M_{H_z}) = ([x_i], X, (M_{H_z})X),$$

where (x^i, x^i_j) are the coordinates of z, and M_{H_z} is the matrix of coordinates of H_z in the Grassmann manifold $\mathcal{G}_n(T_z(\mathcal{F}M))$. □

5. Non-holonomic and semi-holonomic frames of second-order

Now we will recall the definition of the manifolds $\tilde{\mathcal{F}}^2(M)$ and $\hat{\mathcal{F}}^2(M)$ of non-holonomic and semi-holonomic frames of second-order over a real (or complex) manifold M. Then we will prove that the projection $\pi_{\mathcal{H}} : \mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to \mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$, introduced in Section 3, which sends a horizontal n-basis to its corresponding horizontal n-plane, induces another projection $\pi : \tilde{\mathcal{F}}^2(M) \to \hat{\mathcal{F}}^2(M)$, which will become an isomorphic $\mathrm{GL}(n, \mathbb{K})$-principal bundle.

Definition 5.1. If $j^1_{0,z}\varphi$ is the 1-jet of a $\mathbb{K}$-differentiable map $\varphi : (\mathbb{K}^n, 0) \to (\mathcal{F}(M), z)$ such that $\pi^1_0 \circ \varphi$ is a $\mathbb{K}$-diffeomorphism, we will say that $j^1_{0,z}\varphi$ is a non-holonomic frame of second-order at the point $x = \pi^1_0(z) \in M$. If, additionally, $\varphi(0) = z = j^1_{0,x}(\pi^1_0 \circ \varphi)$, then it is said that $j^1_{0,z}\varphi$ is a semi-holonomic frame of second-order at the point $x = \pi^1_0(z) \in M$.

The set $\tilde{\mathcal{F}}^2(M)$ of all non-holonomic frames of second-order on M is the total space of a principal bundle over M, with projection $\tilde{\pi}^2_0 : j^1_{0,z}\varphi \in \tilde{\mathcal{F}}^2(M) \to x = \pi^1_0(z) \in M$ and group

$$\tilde{G}^2(n, \mathbb{K}) = \mathrm{GL}(n, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K}) \times L_2(n, \mathbb{K}),$$

where $L_2(n, \mathbb{K})$ denotes the additive group of $\mathbb{K}$-bilinear maps from $\mathbb{K}^n \times \mathbb{K}^n$ into $\mathbb{K}^n$. Since $\mathfrak{gl}(n, \mathbb{K}) = \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^n)$, we can identify $f \in L_2(n, \mathbb{K})$ with $\bar{f} \in \mathrm{Hom}(\mathbb{K}^n, \mathfrak{gl}(n, \mathbb{K}))$ given by $\bar{f}(w)(v) = f(v, w)$. If $\{E_i\}$ is the usual basis of $\mathbb{K}^n$ and $\{\epsilon^i\}$ its dual basis, then the linear maps $E^i_j = E_i \oplus \epsilon^j$ form the usual basis of $\mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^n) = \mathcal{M}_{n\times n}(\mathbb{K})$. If $\bar{f}(E_k) = f^{i\,j}_{\;k} E^i_j$ and $f(E_j, E_k) = f^i_{jk} E_i$, then $f^{i\,j}_{\;k} = f^i_{jk}$. The product in $\tilde{G}^2(n, \mathbb{K})$ is given by

$$(a, b, f)(a', b', f') = (a \circ a', b \circ b', a \circ f' + f \circ (a' \times b')).$$

If $\mathbb{K} = \mathbb{C}$ then $\tilde{G}^2(n, \mathbb{C})$ is a complex Lie group. Let $\varphi : (\mathbb{K}^n, 0) \to (\mathcal{F}(M), z)$ be a $\mathbb{K}$-differentiable map such that $\pi^1_0 \circ \varphi$ is a $\mathbb{K}$-diffeomorphism. In local coordinates $\varphi(r^a) \equiv (\varphi^i(r^a), \varphi^i_j(r^a))$, hence the coordinates of $j^1_{0,z}\varphi$ are given by

$$\left(\varphi^i(0), \varphi^i_j(0), \frac{\partial \varphi^i}{\partial r^j}(0), \frac{\partial {\varphi^k}_l}{\partial r^j}(0)\right),$$

so in $\tilde{\mathcal{F}}^2(M)$ we have a system of local coordinates of the form $(x^i, x^i_j, y^i_j, {x^{k_l}}_j)$, with (x^i, x^i_j) the coordinates of $z = \varphi(0)$. Furthermore,

$$\varphi_*(0) \cdot \frac{\partial}{\partial r^j}(0) = \frac{\partial \varphi^i}{\partial r^j}(0)\frac{\partial}{\partial x^i}(z) + \frac{\partial {\varphi^k}_l}{\partial r^j}(0)\frac{\partial}{\partial {x^k}_l}(z) = y^i_j \frac{\partial}{\partial x^i}(z) + {x^{k_l}}_j \frac{\partial}{\partial x^{k_l}}(z)$$

with the matrix $[y^i_j]$ non-singular, because $\pi^1_0 \circ \varphi$ is a diffeomorphism.

Let us consider $j^1_{0,z}\varphi \in \tilde{F}^2(M)$. The semi-holonomy condition $\varphi(0) = z = j^1_{0,x}(\pi^1_0 \circ \varphi)$ is characterized in local coordinates by the equality $y^i_j = x^i_j$. Hence, there is in $\hat{\mathcal{F}}^2(M) \subset$

$\tilde{\mathcal{F}}^2(M)$ a system of local coordinates of the form $(x^i, x^i_j, x^{k_l}_j)$. Therefore, $\hat{\mathcal{F}}^2(M)$ is a regular submanifold of $\tilde{\mathcal{F}}^2(M)$. The restriction $\hat{\pi}^2_0$ of $\tilde{\pi}^2_0$ is the principal bundle $\hat{\pi}^2_0 : \hat{\mathcal{F}}^2(M) \to M$ of all semi-holonomic frames of second-order over M. Its structure group is

$$\hat{G}^2(n, \mathbb{K}) = \mathrm{GL}(n, \mathbb{K}) \times L_2(n, \mathbb{K}).$$

It is a closed Lie subgroup of $\tilde{G}^2(n, \mathbb{K})$, and its group law is given by

$$(a, f)(a', f') = (a \circ a', a \circ f' + f \circ (a' \times a')).$$

For $\mathbb{K} = \mathbb{C}$ we have that $\hat{G}^2(n, \mathbb{C})$ is a complex Lie subgroup of $\tilde{G}^2(n, \mathbb{C})$. On the other hand,

$$G^2(n, \mathbb{K}) = \{(a, f) \in \hat{G}^2(n, \mathbb{K});\ f \text{ symmetric}\}$$

is a closed Lie subgroup of $\tilde{G}^2(n, \mathbb{K})$. The elements $j^2_{0,0}h \in G^2(n, \mathbb{K})$ are second-order Taylor polynomials of $\mathbb{K}$-diffeomorphisms $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$. The group law in $G^2(n, \mathbb{K})$ is just the chain rule for first- and second-order differentials at 0 of the composition of two $\mathbb{K}$-diffeomorphisms. Notice that the group law in the larger group $\hat{G}^2(n, \mathbb{K})$ is *formally* the same. If $\mathbb{K} = \mathbb{C}$ then $G^2(n, \mathbb{C})$ is a closed complex Lie subgroup of $\hat{G}^2(n, \mathbb{C})$. The regular submanifold $\mathcal{F}^2(M) \subset \hat{\mathcal{F}}^2(M)$ given by those semi-holonomic frames with $x^{k_l}{}_j = x^{l_k}{}_j$ is a $G^2(n)$-sub-bundle $\pi^2_0 : \mathcal{F}^2(M) \to M$, which is in fact (isomorphic to) the bundle $F^2(M)$ of holonomic frames of second-order on M, given in Section 1 (see [9,16,17,31]).

If $j^1_{0,z}\varphi \in \tilde{\mathcal{F}}^2(M)$, then $\pi^1_0 \circ \varphi$ is a diffeomorphism and $\{\varphi_*(0) \cdot (\partial/\partial r^j)(0),\ j = 1, \dots, n\}$ is a horizontal n-basis. We define a map

$$\tilde{\nu} : j^1_{0,z}\varphi \in \tilde{\mathcal{F}}^2(M) \to \left\{\varphi_*(0) \cdot \frac{\partial}{\partial r^1}(0), \dots, \varphi_*(0) \cdot \frac{\partial}{\partial r^n}(0)\right\} \in \mathcal{S}t_n\mathcal{H}_{\mathcal{F}M}.$$

Identifying $T_0\mathbb{K}^n$ and $\mathbb{K}^n$, we can see $\tilde{\nu}(j^1_{0,z}\varphi)$ as the horizontal n-basis $\{\varphi_*(0)E_1, \dots, \varphi_*(0)E_n\}$.

Since we have seen in Theorem 4.1 that $\hat{H}^2(M)$ and $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$ are also diffeomorphic, we can adapt the diffeomorphism given in [23] between $\hat{\mathcal{F}}^2(M)$ and $\hat{H}^2(M)$ to define a map $\hat{\nu}$ given by

$$\hat{\nu} : j^1_{0,z} \in \varphi\hat{\mathcal{F}}^2(M) \to \left\langle\left\{\varphi_*(0) \cdot \frac{\partial}{\partial r^j}(0),\ j = 1, \dots, n\right\}\right\rangle \in \mathcal{G}_n\mathcal{H}_{\mathcal{F}M}.$$

Identifying $T_0\mathbb{K}^n$ and $\mathbb{K}^n$, we see that $\hat{\nu}(j^1_{0,z}\varphi)$ is the horizontal n-plane $\varphi_*(0)(\mathbb{K}^n) \subset T_z(\mathcal{F}M)$.

Theorem 5.2. *The maps* $\pi = \hat{\nu}^{-1} \circ \pi_{\mathcal{H}} \circ \tilde{\nu} : \tilde{\mathcal{F}}^2(M) \to \hat{\mathcal{F}}^2(M)$ *and* $\pi_{\mathcal{H}} : \mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to \mathcal{G}_n\mathcal{H}_{\mathcal{F}(M)}$ *are isomorphic* $\mathrm{GL}(n, \mathbb{K})$*-principal bundles, via the bundle map*

$$(\tilde{\nu}, \hat{\nu}) : (\tilde{\mathcal{F}}^2M, \hat{\mathcal{F}}^2M) \to (\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M}, \mathcal{G}_n\mathcal{H}_{\mathcal{F}M}).$$

Proof. If we consider in the local coordinates induced by M in $\tilde{\mathcal{F}}^2(M)$, $\hat{\mathcal{F}}^2(M)$, $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M}$ and $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$, then we have that, locally, the map $\tilde{\nu}$ is the identity (reflecting that $\tilde{\mathcal{F}}^2(M)$

and $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M}$ are essentially identical) whereas $\hat{\nu}$ is given by

$$\hat{\nu}([x^i], X, [x^{ij}{}_k]) = ([x_i], X, [x^{ij}{}_k]X^{-1}).$$

Thus, both $\tilde{\nu}$ and $\hat{\nu}$ are diffeomorphisms. The action of $\mathrm{GL}(n, \mathbb{K})$ on $\tilde{\mathcal{F}}^2(M)$ is defined as follows: if $L \in \mathrm{GL}(n, \mathbb{K})$ and $j^1_{0,z}\varphi \in \tilde{\mathcal{F}}^2(M)$, we define

$$(j^1_{0,z}\varphi)L = j^1_{0,z}(\varphi \circ L).$$

It is evident that the action is free. Let $\pi = \hat{\nu}^{-1} \circ \pi_{\mathcal{H}} \circ \tilde{\nu}$. The map $\pi : \tilde{\mathcal{F}}^2(M) \to \hat{\mathcal{F}}^2(M)$ defines a structure of $\mathrm{GL}(n, \mathbb{K})$-principal bundle. If L^i_j is the matrix of L in the usual basis, then

$$(\varphi \circ L)_*(0) \cdot \frac{\partial}{\partial r^j}(0) = \frac{\partial \varphi^i}{\partial r^s}(0)L^s_j\frac{\partial}{\partial x^i}(z) + \frac{\partial \varphi^k{}_l}{\partial r^s}(0)L^s_j\frac{\partial}{\partial x^k{}_l}(z),$$

hence, $\tilde{\nu}((j^1_{0,z}\varphi)L) = (\tilde{\nu}(j^1_{0,z}\varphi))L$. □

As a consequence, we can see a non-holonomic frame of second-order at $x \in M$ as a horizontal tangent n-basis at some z in the fiber $(\pi^1_0)^{-1}(x) \subset F(M)$ and π projects all horizontal tangent n-bases generating the same horizontal n-space H_z on the same semi-holonomic frame of second-order at $x \in M$, which can be identified with H_z. Geometrically, the first copy of $\mathrm{GL}(n, \mathbb{K})$ in $\tilde{G}^2(n, \mathbb{K})$ moves non-holonomic frames along that fiber, whereas the second copy just changes horizontal basis at a particular horizontal space. The action of $L_2(n, \mathbb{K})$ is explained in the next section.

6. The bundles $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}(M)}$ and $\mathcal{G}_n\mathcal{H}_{\mathcal{F}(M)}$ are isomorphic to $\tilde{\mathcal{F}}^2(M)$ and $\hat{\mathcal{F}}^2(M)$

In [23] it is proved that the projection

$$\pi^{\hat{H}}_1 : \hat{u}(H_z) \in \hat{H}^2(M) \to z \in \mathcal{F}(M)$$

is a $L_2(n, \mathbb{K})$-principal bundle. Let us consider the analogous projection

$$\pi^{\mathcal{GH}}_1 : H_z \in \mathcal{G}_n\mathcal{H}_{\mathcal{F}M} \to z \in \mathcal{F}(M).$$

Theorem 6.1. *The map $\pi^{\mathcal{GH}}_1 : \mathcal{G}_n\mathcal{H}_{\mathcal{F}M} \to \mathcal{F}(M)$ defines a principal bundle with structure group $L_2(n, \mathbb{K})$. Moreover, it is isomorphic to the $L_2(n, \mathbb{K})$-principal bundles $\pi^{\hat{H}}_1 : \hat{H}^2(M) \to \mathcal{F}(M)$ and $\hat{\pi}^2_1 : \hat{\mathcal{F}}^2(M) \to \mathcal{F}(M)$.*

Proof. The action of $L_2(n, \mathbb{K})$ on $\mathcal{G}_n\hat{\mathcal{H}}_{\mathcal{F}(M)}$ is defined as follows: given $H_z \in \mathcal{G}_n\hat{\mathcal{H}}_{\mathcal{F}(M)}$ and $f \in \mathrm{Hom}(\mathbb{K}^n, \mathfrak{gl}(\mathfrak{n}, \mathbb{K}))$, then $(H_z)f = H'_z$ is the horizontal space at z spanned by the vectors $\theta^{-1}_{/H_z}(E_j) + f(E_j)^*_z$, with $j = 1, \dots, n$. This action of $L_2(n, \mathbb{K})$ is clearly free. Furthermore, its transivity on the fiber appears in the study of Bernard's structure tensor of a G-structure on M, [2], see also [14, p. 41], where the linear map $f \in \mathrm{Hom}(\mathbb{K}^n, \mathfrak{gl}(\mathfrak{n}, \mathbb{K}))$

such that $(H_z)f = H'_z$ is denoted by $S_{H,H'}$. The action of $L_2(n,\mathbb{K})$ on $\hat{H}^2(M)$ is given by $\hat{u}(H_z)f = \hat{u}(H'_z)$. On the other hand, $L_2(n,\mathbb{K})$ also acts on $\hat{\mathcal{F}}^2(M)$: if $([x^i],[x^i_j],[x^{k_l}_j])$ are the coordinates of $j^1_{0,z}\varphi \in \hat{\mathcal{F}}^2(M)$ let us put $X = [x^i_j]$ and identify $\bar{\xi} = [x^{i_j}_k] \in \mathcal{M}_{n^2\times n}(\mathbb{K}) \equiv \mathrm{Hom}(\mathbb{K}^n, \mathfrak{gl}(\mathfrak{n},\mathbb{K}))$ with its corresponding bilinear map $\xi \in L_2(n,\mathbb{K})$ with components $[x^i_{jk}]$. Then the action of $f \in L_2(n,\mathbb{K}) \equiv (\mathrm{id}_{\mathbb{K}^n}, f) \in \hat{G}^2(n,\mathbb{K})$ is given in coordinates by

$$(j^1_{0,z}\varphi)f \equiv ([x^i], X, \xi + X \circ f).$$

In this way, the projection $\pi^{\hat{\mathcal{F}}}_1 : \hat{\mathcal{F}}^2(M) \to \mathcal{F}(M)$ sending a semi-holonomic frame $j^1_{0,z}\varphi$ at $x = \pi^1_0(z)$ to the linear frame z defines a principal bundle with structure group $L_2(n,\mathbb{K})$. The map

$$\hat{u} : \mathcal{G}_n\mathcal{H}_{\mathcal{F}M} \to \hat{H}^2(M)$$

defined in Section 4 becomes an isomorphism of $L_2(n,\mathbb{K})$-principal bundles over $\mathcal{F}(M)$. Moreover, the diffeomorphism

$$\hat{u} \circ \hat{\nu} : \hat{\mathcal{F}}^2(M) \to \hat{H}^2(M)$$

is a bundle isomorphism too. In fact, in local coordinates, we have that $\hat{u} \circ \hat{\nu}$ is the identity (it is just the map denoted as u in [23]). □

Theorem 6.2. *The bundles $\mathcal{S}t_n\mathcal{H}_{\mathcal{F}M} \to M$ and $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M} \to M$ are, respectively, isomorphic to the bundles $\tilde{\mathcal{F}}^2(M) \to M$ and $\hat{\mathcal{F}}^2(M) \to M$ of non-holonomic and semi-holonomic frames of second-order on M.*

Proof. If $\pi_P : P \to M$ is a G-principal bundle and $F : Q \to P$ is a diffeomorphism, then, the map $\pi_Q : Q \to M$ defined by $\pi_Q = \pi_P \circ F$ is a G-principal bundle isomorphic to $\pi_P : P \to M$ with the action $Q \times G \to Q$ given by $qg = F^{-1}(F(q)g)$. Our statement follows from this fact and from Theorems 5.2 and 6.1. □

7. Linear connections as reductions of $\hat{\mathcal{F}}^2(M)$

Here we will give a new proof of a theorem of Libermann [24], that characterized linear connections on a real manifold. For torsionless connections it was previously found by Kobayashi [16] (see also [18]), de León and Ortacgil gave in [23] another proof using their bundle $\hat{H}^2(M) \subset \mathcal{F}(\mathcal{F}M)$. Ours will be moulded in the same idea, but we will instead use the isomorphic bundle $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$ of horizontal n-planes on $\mathcal{F}(M)$. With it the theorem becomes almost a tautology. In order to assure that the proof will also work for complex data, let us point out that if M is a complex manifold, G a complex Lie group and H a complex closed Lie subgroup of G then G/H is a complex G-homogeneous manifold and the following fact is still true: each holomorphic H-reduction of a holomorphic principal G-bundle $\pi : P \to M$ determines in a bijective way a holomorphic global section $\sigma : M \to E$ of the associated complex bundle $E = P \times_G (G/H) = P/H$.

Theorem 7.1. *There exists a bijective correspondence between the set of* $\mathbb{K}$*-differentiable linear connections of a real or complex manifold M and the set of* $\mathrm{GL}(n,\mathbb{K})$*-reductions of its bundle* $\hat{\mathcal{F}}^2(M)$ *of semi-holonomic frames of second-order.*

Proof. Let us recall that a $\mathbb{K}$-differentiable linear connection on M is a $\mathbb{K}$-differentiable distribution $\mathcal{D}$ on $\mathcal{F}(M)$ (that is to say, $\mathcal{C}^\infty$ if M is real, holomorphic if M is complex) such that, for every $z \in \mathcal{F}(M)$, $\mathcal{D}_z$ is a horizontal n-plane and $\mathcal{D}$ is invariant under the action of the group $\mathrm{GL}(n,\mathbb{K})$. Therefore, $Q = \{\mathcal{D}_z; z \in \mathcal{F}(M)\}$ is a subset of the $\hat{G}^2(n,\mathbb{K})$-principal bundle $\mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$ of horizontal n-planes on $\mathcal{F}(M)$, which is isomorphic to $\hat{\mathcal{F}}^2(M)$, by Theorem 6.2. Let us see that Q is a $\mathrm{GL}(n,\mathbb{K})$-reduction. It is clear that Q is invariant under the action of the closed subgroup $\mathrm{GL}(n,\mathbb{K})$. The differentiability of $\mathcal{D}$ means that for each $z_0 \in \mathcal{F}(M)$ there exists an open neighborhood V of z_0 in $\mathcal{F}(M)$ and n independent horizontal $\mathbb{K}$-differentiable vector fields $X_1,\dots,X_n$ on V such that, for all $z \in V$, we have $\mathcal{D}_z = \langle X_1(z),\dots,X_n(z)\rangle$. Thus, $z \in V \to \mathcal{D}_z \in \mathcal{G}_n\mathcal{H}_{\mathcal{F}M}$ is a $\mathbb{K}$-differentiable local section taking values in Q. It follows that Q is a $\mathrm{GL}(n,\mathbb{K})$-reduction, since we can use Lemma 1 in Chapter II of [19] (which is valid for real and complex data, because its proof depends on the theorem of Frobenius, true in both cases). The converse is obvious. □

It is well-known that neither analytic connections on an analytic real manifold nor holomorphic connections on a complex manifold M need to exist. A necessary and sufficient condition for their existence was given by Atiyah [1]. Stein manifolds and complex Lie groups admit holomorphic connections. If an analytic connection exists on a complex manifold M, Theorem 7.1 just says that it can be seen as a $\mathrm{GL}(n,\mathbb{C})$-sub-bundle of $\hat{\mathcal{F}}^2(M)$.

8. $\tilde{\mathcal{F}}^2(M)$ and $\hat{\mathcal{F}}^2(M)$ as extensions of $F^2(M)$

Let us denote $e_1 = j^1_{0,0}(\mathrm{id}_{\mathbb{K}^n})$. It is well-known that the group $\tilde{G}^2(n,\mathbb{K})$, described in Section 5, is isomorphic to the group of 1-jets of the form $j_{e_1,\tilde{\Psi}e_1}\tilde{\Psi}$, where $\tilde{\Psi} : \mathcal{F}(\mathbb{K}^n) \to \mathcal{F}(\mathbb{K}^n)$ is a principal bundle isomorphism inducing the identity on $\mathrm{GL}(n,\mathbb{K})$, and such that its underlying map $\Psi : \mathbb{K}^n \to \mathbb{K}^n$ satisfies $\Psi(0) = 0$ [12]. Analogously, the group $\hat{G}^2(n,\mathbb{K})$ is isomorphic to the subgroup of $\tilde{G}^2(n,\mathbb{K})$ given by

$$\{j^1_{e_1,\tilde{\Psi}e_1}\tilde{\Psi} \in \tilde{G}^2(n,\mathbb{K}); \tilde{\Psi}(e_1) = j^1_{0,0}\Psi\}.$$

Let $\tilde{\Psi} : (\mathcal{F}(\mathbb{K}^n), e_1) \to (\mathcal{F}(M), z)$ be a principal bundle isomorphism, and let $\Psi : (\mathbb{K}^n, 0) \to (M, x)$ be the underlying map induced by $\tilde{\Psi}$. The 1-jet $j^1_{e_1,z}\tilde{\Psi}$ can be identified with the tangent map

$$\tilde{\Psi}_*(e_1) : T_{e_1}(\mathcal{F}\mathbb{K}^n) \equiv \mathbb{K}^n \oplus \mathfrak{gl}(n,\mathbb{K}) \to T_z(\mathcal{F}M).$$

It is well-known that such kind of $\mathbb{K}$-linear frames of the manifold $\mathcal{F}(M)$ are just the non-holonomic frames of second-order at the point $x = \Psi(0) \in M$. If $z = \tilde{\Psi}(e_1) = j^1_{0,\Psi(0)}\Psi$, then the frame is semi-holonomic. The resulting non-holonomic and semi-holonomic frame bundles will be denoted by $\tilde{F}^2(M)$ and $\hat{F}^2(M)$. Of course, $\tilde{F}^2(M)$ and

$\hat{F}^2(M)$ are, respectively, isomorphic to $\tilde{\mathcal{F}}^2(M)$ and $\hat{\mathcal{F}}^2(M)$. If $\Psi : (M, x) \to (N, y)$ is a (local) diffeomorphism, we will denote by

$$F^{(1)}\Psi : j^1_{0,x} f \in \mathcal{F}(M) \to j^1_{0,y}(\Psi \circ f) \in \mathcal{F}(N)$$

its prolonged map between the bundle of $\mathbb{K}$-linear frames. If $j^1_{e_1,\tilde{\Psi}(e_1)}\tilde{\Psi}$ is a semi-holonomic frame of second-order and $\tilde{\Psi} = F^{(1)}\Psi$, we will say that the frame is a holonomic frame of second-order or, simply, a frame of second-order on M. The reason for this terminology is that this bundle $\pi^2_0 : F^2(M) \to M$ of second-order frames over M is a principal bundle of group

$$\{j^1_{e_1,\tilde{\Psi}(e_1)}\tilde{\Psi} \in \hat{G}^2(n, \mathbb{K}); \tilde{\Psi} = F^{(1)}\Psi\}$$

which is isomorphic to $G^2(n, \mathbb{K})$. Moreover, $F^2(M)$ and $\mathcal{F}^2(M)$ are isomorphic as $G^2(n, \mathbb{K})$-principal bundles over M. From this point of view, the product in $\tilde{G}^2(n, \mathbb{K})$, $\hat{G}^2(n, \mathbb{K})$ and $G^2(n, \mathbb{K})$ as well as the respective actions of these groups on $\tilde{F}^2(M)$, $\hat{F}^2(M)$ and $F^2(M)$ are given by composition of jets. In local coordinates, an element $(x^i, x^i_j, y^i_j, x^i_{jk})$ of $\tilde{F}^2(M)$ belongs to $F^2(M)$ if and only if $y^i_j = x^i_j$ and $x^i_{jk} = x^i_{kj}$.

Let us consider now the manifold $F^2(M) \times \tilde{G}^2(n, \mathbb{K})$. If $p = j^1_{e_1,\tilde{\Psi}(e_1)}\tilde{\Psi} \in F^2(M)$ and $k = j^1_{e_1,\tilde{\Phi}(e_1)}\tilde{\Phi} \in \tilde{G}^2(n, \mathbb{K})$, then, we will denote by $pk = j^1_{e_1,\tilde{\Psi}\circ\tilde{\Phi}(e_1)}(\tilde{\Psi} \circ \tilde{\Phi})$ the restriction of the action of $\tilde{G}^2(n, \mathbb{K})$ on $\tilde{F}^2(M)$.

Definition 8.1. Let $\pi : P \to M$ be a (holomorphic) differentiable principal bundle with (complex) structure Lie group G, and let us suppose that G is a (complex) Lie subgroup of a (complex) Lie group K. Let $P^K = P \times_G K = P \times K/\sim$ be the quotient (complex) manifold obtained from $P \times K$ and the equivalence relation

$$(p, k) \sim (p', k') \Leftrightarrow \text{there exists } a \in G \text{ such that } p' = pa, k' = a^{-1}k.$$

We will denote as $[(p, k)]$ the equivalence class of (p, k) and by $\mathrm{pr} : (p, k) \in P \times K \to [(p, k)] \in P^K$ the quotient map. The associated bundle

$$\pi^K : [(p, k)] \in P^K \to \pi(p) \in M$$

is known as the extension of P by the group K.

It is immediate that $\pi^K : P^K \to M$ is a (holomorphic) K-principal bundle. Let us remark that if $A \subset M$ is open and $\tau : p \in \pi^{-1}(A) \to (x, g) \in A \times G$ is a local trivialization of P, then $\tau^K : [(p, k)] \in (\pi^K)^{-1}(A) \to (x, gk) \in A \times K$ is a local trivialization of P^K.

Theorem 8.2. (a) *The map* $\vartheta : [(p, k)] \in F^2(M)^{\tilde{G}^2(n,\mathbb{K})} \to pk \in \tilde{F}^2(M)$ *is an isomorphism of* $\tilde{G}^2(n, \mathbb{K})$*-principal bundles.* (b) *The map* $\vartheta : [(p, k)] \in F^2(M)^{\hat{G}^2(n,\mathbb{K})} \to pk \in \hat{F}^2(M)$ *is an isomorphism of* $\hat{G}^2(n, \mathbb{K})$*-principal bundles.*

Proof. (a) First of all, ϑ is well defined: if $(p, k), (p', k') \in F^2(M) \times \tilde{G}^2(n, \mathbb{K})$ and $a \in G^2(n, \mathbb{K})$ verify $p' = pa$ and $k' = a^{-1}k$, then $p'k' = paa^{-1}k = pk$. Let us take

$(p, k), (p', k') \in F^2(M) \times \tilde{G}^2(n, \mathbb{K})$ such that $pk = p'k'$. In order to prove that ϑ is injective, we must find $a \in G^2(n, \mathbb{K})$ such that $p' = pa$ and $k' = a^{-1}k$. It is enough to prove that $a = k(k')^{-1} \in G^2(n, \mathbb{K})$, since $pk = p'k'$ if and only if $pk(k')^{-1} = p'$. Moreover, it is enough to show that if $p, p' \in F^2(M)$ and $h \in \tilde{G}^2(n, \mathbb{K})$ verify $p' = ph$, then $h \in G^2(n, \mathbb{K})$, or, what is the same, that p and p' lie in the same fiber of $F^2(M) \to M$. But p and p' lie in the same fiber of $\tilde{F}^2(M) \to M$, and that follows from the fact that the action of $\tilde{G}^2(n, \mathbb{K})$ on $\tilde{F}^2(M)$ is free. Next, for $p = j^1_{e_1, \tilde{\Psi}(e_1)} \tilde{\Psi} \in F^2(M)$, let $F^{(1)}\Psi : \mathcal{F}(\mathbb{K}^n) \to \mathcal{F}(M)$ be the prolonged map of the induced map $\Psi : \mathbb{K}^n \to M$. Since $F^{(1)}\Psi$ also induces Ψ, then $q = j^1_{e_1, F^{(1)}\Psi(e_1)} F^{(1)}\Psi \in F^2(M)$ and p lie in the same fiber of $\tilde{F}^2(M) \to M$. Thus, there exists $k \in \tilde{G}^2(n, \mathbb{K})$ such that $p = qk$ and $\vartheta([(q, k)]) = p$. Hence, ϑ is surjective. On the other hand, it is immediate that $\vartheta([(p, kk')]) = \vartheta([(p, k)])k'$ for $p \in F^2(M)$ and $k, k' \in \tilde{G}^2(n, \mathbb{K})$. Moreover, if $i : F^2(M) \times \tilde{G}^2(n, \mathbb{K}) \to \tilde{F}^2(M) \times \tilde{G}^2(n, \mathbb{K})$ denotes the inclusion and $R : \tilde{F}^2(M) \times \tilde{G}^2(n, \mathbb{K}) \to \tilde{F}^2(M)$ denotes the action, it follows that ϑ is differentiable, since $\vartheta \circ \mathrm{pr} = R \circ i$ and pr is a submersion. Finally, let us prove that, in local coordinates, ϑ is the identity. If $(p, k) \in F^2(M) \times \tilde{G}^2(n, \mathbb{K})$, with $x = \pi^2_0(p)$ and $k = (a^i_j, b^i_j, c^i_{jk})$, we can consider p as an element of $\tilde{F}^2(M)$ whose coordinates $(x^i, x^i_j, y^i_j, x^i_{jk})$ satisfy $x^i_j = y^i_j$ and $x^i_{jk} = x^i_{kj}$. In the trivialization induced by a chart of M, the fiber part $[(p, k)]$ has coordinates $(x^i_j, x^i_j, x^i_{jk})(a^i_j, b^i_j, c^i_{jk})$, by virtue of the remark after Definition 8.1. The product at the second component is the product in $\tilde{G}^2(n, \mathbb{K})$. Furthermore, if $p = j^1_{e_1, \tilde{\Psi}(e_1)} \tilde{\Psi} \in F^2(M)$ and $k = j^1_{e_1, \tilde{\Phi}(e_1)} \tilde{\Phi} \in \tilde{G}^2(n, \mathbb{K})$, then the coordinates of $pk = j^1_{e_1, \tilde{\Psi}(\tilde{\Phi}(e_1))} (\tilde{\Psi} \circ \tilde{\Phi})$ are the previous ones, since $\tilde{\pi}^2_0(pk) = \tilde{\pi}^2_0(p) = \Psi(0)$ and the product of $\tilde{G}^2(n, \mathbb{K})$ is given by jet composition. So, in coordinates, ϑ is the identity and, therefore, ϑ is an isomorphism of bundles. The proof of (b) is completely analogous. □

Let us point out that Theorem 8.2 allows us to give a different proof of Libermann's Theorem 7.1 as follows: since $G^1(n, \mathbb{K}) = \mathrm{GL}(n, \mathbb{K})$ is a (complex when $\mathbb{K} = \mathbb{C}$) closed Lie subgroup of $\hat{G}^2(n, \mathbb{K})$, it follows that each $G^1(n, \mathbb{K})$-reduction of $\hat{F}^2(M) = F^2(M)^{\hat{G}^2(n,\mathbb{K})}$ determines a global section $\Gamma : M \to E$ (holomorphic when $\mathbb{K} = \mathbb{C}$ and M is complex) where E is the associated bundle

$$E = F^2(M)^{\hat{G}^2(n,\mathbb{K})} \times_{\hat{G}^2(n,\mathbb{K})} \hat{G}^2(n, \mathbb{K})/G^1(n, \mathbb{K}).$$

If we consider the trivializations of E associated to an atlas on M and we read Γ through them, we obtain that the fiber part of $\Gamma(x)$ behaves under change of coordinates as (bilinear maps associated to) Christoffel's symbols at x do, determining therefore a linear connection on M. Conversely, each (holomorphic) linear connection on a (complex) manifold M gives rise, via its (holomorphic) Christoffel's symbols, to a (holomorphic) global section $\Gamma : M \to E$ and therefore to a $G^1(n, \mathbb{K}) = \mathrm{GL}(n, \mathbb{K})$-reduction of $\hat{F}^2(M)$.

In fact, this argument is the *extension* of that used by Kobayashi [16,18] to prove that torsionless connections on a real manifold are in bijective correspondence with the $\mathrm{GL}(n, \mathbb{R})$-reductions of $F^2(M)$ [16,18]. Christoffel's symbols symmetry in the torsionless case is due to symmetry of the bilinear part of the elements in the smaller group $G^2(n, \mathbb{R})$.

Finally, let us say that bundles of non-holonomic and semi-holonomic higher order frames can be defined inductively [11,24,31]. The isomorphism given by Theorem 8.2 can be generalized in that way to higher order bundles, since the analogous maps in (a) and (b) are naturally defined at any order.

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